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TECHNICAL NOTE 3152

TRANSVERSE OSCILLATIONS IN A CYLINDRICAL COMBUSTION CHAMBER

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Cleveland, Ohio



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Page 8: Equation (13) should be

$$\frac{-4(R^2 - r_0^2)}{\pi^2 r_1 Ka_f^2} \text{Im}(\omega) = \frac{z^2 \left(1 - \frac{z_0^2}{z^2}\right) \left[J_n(z_1) - N_n(z_1) \frac{J'_n(z)}{N'_n(z)} \right]^2}{\frac{1 - \left(\frac{n}{z}\right)^2}{N_n'^2(z)} - \frac{1 - \left(\frac{n}{z_0}\right)^2}{N_n'^2(z_0)}} \equiv A_1$$

Figure 4, which depends on equation (13), is correct as it stands, however.

Page 8: The left side of equation (14) should be

$$\frac{-4R^2}{\pi^2 r_1 Ka_f^2} \text{Im}(\omega)$$

Pages 9 and 10: The left sides of equations (16) and (18) should be

$$\frac{-4(R^2 - r_0^2)}{\pi^2 r_1 Ka_f^2} \text{Im}(\omega)$$



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SUMMARY

A study has been made of certain features of transverse resonance in an idealized combustion chamber. Modes of weak oscillation are described, with average Mach number neglected but with the effect of axial gradient of mean temperature considered. These weak oscillations are assumed to be amplified by coupling with the vigorous combustion occurring in flame-holder wakes.

A flame-holder wake is idealized as a thin annulus performing mechanical work on the surrounding gases. The amplification of the various modes depends on the flame-holder diameter, centerbody diameter, and on time-lag effects.

The nature of finite transverse periodic waves is also analyzed. Results show that such waves have frequencies independent of amplitude and do not steepen with time, at least to second order.

INTRODUCTION

The flow in the combustion chambers of jet engines is frequently subject to a resonant oscillation, the character of which depends on the particular engine configuration. The transverse-wave motions described by the classical theory of acoustic oscillations inside cylindrical enclosures (refs. 1 and 2) may sometimes be important.

In the classical theory, the fluid medium is considered to be at rest and in a uniform state except for weak isoenergetic fluctuations in velocity and state properties. Under these assumptions, the motion is governed by the wave equation, written here in cylindrical coordinates for the pressure disturbance (sketch (a))

$$p'_{tt} = a^2 \nabla^2 p' = a^2 \left(p'_{xx} + p'_{rr} + \frac{1}{r} p'_r + \frac{1}{r^2} p'_{\theta\theta} \right) \quad (1a)$$

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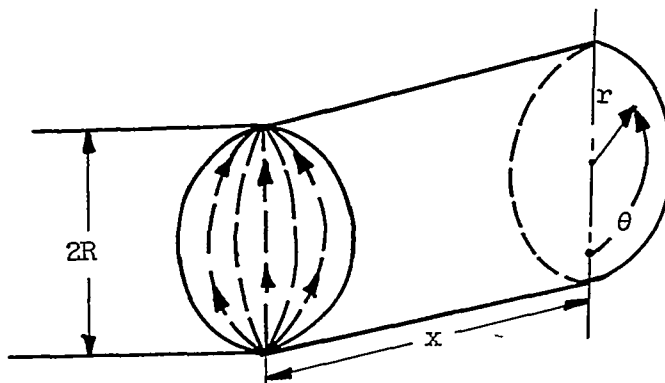
Velocity and pressure fluctuations are related by the equation

$$\bar{\rho} \frac{d\mathbf{q}}{dt} + \nabla p' = 0 \quad (1b)$$

A class of solutions of equation (1a) suitable for internal flows is

$$p' = e^{i\left(\omega t + n\theta + \sqrt{\frac{\omega^2}{a^2} - \frac{C^2}{R^2}} x\right)} J_n\left(C \frac{r}{R}\right) \quad (2)$$

where n is any positive integer.



(a) Coordinate system and sketch of fundamental sloshing mode.

(A complete list of symbols is given in the appendix.) For resonant oscillations, C and ω are determined by conditions at the ends of the cylinder and from the condition of zero radial velocity at the cylinder wall ($r = R$)

$$J'_n(C) = 0 \quad (3)$$

For each value of n there exists an infinite number of solutions of equation (3) for C , because J_n is an oscillatory function of its argument. Hereinafter, the number m will be used to specify that C is chosen as the $(m + 1)^{\text{th}}$ number for which the derivative of J_n vanishes, but J_n does not.

If consideration is restricted to oscillations independent of x , the two simplest cases are: first "radial" mode ($n = 0$, $m = 1$),

$$p' = e^{3.83i\frac{a}{R}t} J_0\left(3.83 \frac{r}{R}\right) \quad (4)$$

and first "sloshing" mode ($n = 1, m = 0$),

$$p' = e^{i\left(1.84\frac{a}{R}t + \theta\right)} J_1\left(1.84\frac{r}{R}\right) \quad (5)$$

The loci of disturbance motions for the first sloshing mode ($n = 1, m = 0$) are shown in sketch (a). There are two nodes, diametrically opposed, and the fluid appears to slosh between these nodes in the illustrated case of a standing wave. This mode has the lowest frequency ($\frac{\omega R}{a} = 1.84$) of all the possible transverse cylindrical waves.

In the present report, it will be assumed that resonant combustion-chamber oscillations result from the amplification of initially weak transverse pressure disturbances interacting with the combustion process. (The amplification of transverse waves in solid-propellant rockets has been studied by Grad (ref. 3) and Cheng (ref. 4).) On this assumption, the following problems are investigated:

(1) Modes of oscillation: The transverse modes of oscillation should be related to conditions in a combustion chamber, which contains gases in motion with severe gradients of state properties owing to heat released by the combustion process. These conditions are at variance with the assumptions of classical acoustic theory.

(2) Amplification at a thin annulus: Flame holders (fig. 1) induce especially vigorous burning in their wakes. Flame-holder configurations would therefore be expected to affect the amplification of combustor oscillations in the various modes. Amplification is also affected by any time lag in the interaction of the burning process with the oscillations.

(3) Finite transverse waves: Combustion-chamber oscillations may have extremely high amplitudes. The character of finite-strength transverse waves therefore merits an investigation to determine, for example, whether or not compressive transverse waves steepen to form shocks, as do plane waves.

MODES OF OSCILLATION

Differential Equations and Assumptions

The action of a flame holder in a combustor (fig. 1) tends to produce a region extending downstream in which burning is especially vigorous. The gases surrounding the flame-holder wake would presumably be burning, but at a slower rate. These surrounding gases (shown shaded in fig. 1) are assumed to obey the equations of compressible nonviscous flow with heat addition

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$$\left. \begin{aligned}
 \text{Momentum:} \quad & \rho \frac{D\mathbf{q}}{Dt} = -\nabla p \\
 \text{Continuity:} \quad & \rho_t + \nabla \cdot (\rho \mathbf{q}) = 0 \\
 \text{Energy:} \quad & \rho \frac{Dc_v T}{Dt} + p \nabla \cdot \mathbf{q} = \rho H \\
 \text{State:} \quad & p = (\gamma - 1) \rho c_v T
 \end{aligned} \right\} \quad (6)$$

In the absence of disturbance, one-dimensional steady flow is assumed, and equations (6) become

$$\left. \begin{aligned}
 \overline{\rho u u_x} &= -\overline{p_x} \\
 \overline{\rho u} &= \text{constant} \\
 \overline{\rho u c_v T_x} + \overline{p u_x} &= \overline{\rho H} \\
 \overline{p} &= (\gamma - 1) \overline{\rho c_v T}
 \end{aligned} \right\} \quad (7)$$

The disturbance field is assumed to involve the high-frequency motion usual in cases of acoustic resonance. The consequent assumption

$$\frac{\partial}{\partial t} = \text{order} \left(\overline{a} \frac{\partial}{\partial x} \right) \quad (8a)$$

is made concerning derivatives in the disturbance field. Combustors usually produce large temperature changes at rather low Mach number. Accordingly, the assumption of small Mach number is made

$$\overline{M} \ll 1 \quad (8b)$$

The various flow quantities may now be assumed to be perturbed slightly from their steady values; thus, for example,

$$p(x, r, \theta, t) = \overline{p}(x) + p'(x, r, \theta, t) \quad (8c)$$

If equation (8c) and similar expressions for the other flow properties are substituted into equation (6), taking note of equations (7), (8a), and (8b), there follows, to first order,

$$\frac{p'_{tt}}{-2} - \frac{\overline{T}_x}{\overline{T}} p'_x - \nabla^2 p' = \frac{\overline{p}}{c_p \overline{T}} H'_t$$

If it is further assumed that, when the temperature undergoes a relative change, the rate of heat release H has a relative change of the same order, then the right side of the foregoing equation is of order \bar{M} , and, for consistency with equation (8b), should be neglected. The equation replacing the wave equation (1a) is thus

$$\frac{p'_{tt}}{a^2} - \frac{\bar{T}_x}{\bar{T}} p'_x = \nabla^2 p' \quad (9a)$$

To the same order, the following additional equations apply:

$$\bar{\rho} q'_t = -\nabla p' \quad (9b)$$

$$\frac{p'_t}{\bar{p}} = \gamma \frac{\rho'_t}{\bar{\rho}} + \gamma u' \frac{\bar{\rho}_x}{\bar{\rho}} \quad (9c)$$

$$\frac{p'}{\bar{p}} = \frac{\rho'}{\bar{\rho}} + \frac{T'}{\bar{T}} \quad (9d)$$

Thus, to zero order in Mach number, the oscillations are not amplified by coupling between an oscillation and the combustion process. However, to this order in Mach number, the mode of oscillation is affected by the mean temperature gradient \bar{T}_x produced by the burning.

Solutions

Equation (9a) may be satisfied by solutions in the following form (obtained by separation of variables):

$$\frac{p'}{\bar{p}} = G(x) e^{i(\omega t + n\theta)} \left[\alpha J_n \left(C \frac{r}{R} \right) + \beta N_n \left(C \frac{r}{R} \right) \right] \quad (10a)$$

where $G(x)$ must satisfy the equation

$$G''(x) + \frac{\bar{T}_x}{\bar{T}} G'(x) + \left[\frac{\omega^2}{a^2(x)} - \frac{C^2}{R^2} \right] G(x) = 0 \quad (10b)$$

If the distance required for unit relative change in temperature is long compared with the burner diameter, then the term of equation (10b) containing the factor \bar{T}_x may be neglected, and equations (10) may be replaced by the classical solution (eq. (2)), the speed of sound

$\bar{a} = \sqrt{(\gamma-1)c_p \bar{T}}$ being regarded as variable with x . If the term involving \bar{T}_x is not neglected, then equation (10b) must usually be integrated numerically.

The constant C is determined by transverse boundary conditions, and ω is then determined by an axial boundary condition, which is assumed for this case to be

$$\omega = C \frac{a_f}{R} \quad (10c)$$

where a_f is the final (maximum) speed of sound attained in the combustor. If equation (10c) applies, the x -dependence of the oscillation is governed by the equation (which replaces eq. (10b))

$$G'' + \frac{\bar{T}_x}{\bar{T}} G' + \frac{C^2}{R^2} \left[\frac{a_f^2}{\bar{a}(x)^2} - 1 \right] G = 0 \quad (10d)$$

Clearly, if \bar{T} approaches a constant value as x increases, equation (10d) provides that G must become and remain constant (G being assumed to remain finite). Therefore, the effect of equation (10c) is to assume that any oscillation becomes entirely transverse for large x . In the region upstream of the flame holder there is a sinusoidal variation of pressure with x , which for a traveling wave results in a spiral wave form.

Figure 2 illustrates the axial form of the mode corresponding to equation (10c) obtained by numerical integration of equation (10d) for the illustrated axial variation of sound speed and for the ratios $R/L = 0.1$ and 1 . The constant C was taken to correspond to the first sloshing mode ($n = 1, m = 0$), and the solutions were adjusted so that $G(\infty) = 1$ in each case. The larger value of R/L corresponds most closely to a large combustor, while the other value would be more reasonable for a combustor with a very small diameter.

AMPLIFICATION AT A THIN ANNULUS

Assumptions and Boundary Conditions

It is assumed that burning in the wake behind a flame holder, here assumed to be annular, is substantially more vigorous than in the surrounding gases (fig. 1). Thus, the assumptions leading to equations (10) are inappropriate to oscillations occurring within the annulus. A different simplifying assumption is made concerning the annulus; namely, that the region of vigorous burning is extremely thin - too thin to support a transverse pressure difference. Thus, employing subscripts c and w to denote the disturbance fields between centerbody and annulus, and annulus and outer wall, respectively,

$$p'_c(r_1, \theta, t) = p'_w(r_1, \theta, t) \quad (11a)$$

Within the annulus, the burning rate is assumed to undergo relative change in proportion to relative change in temperature (and hence pressure, for sufficiently small axial steady-state gradients (eq. (9))). This fluctuating energy release in the annulus is communicated to the surrounding gases as fluctuating mechanical work, such as would be performed by an aneroid expanding and contracting. The mechanical work would be in proportion to the radial velocity difference

$$v'_w(r_1, \theta, t) - v'_c(r_1, \theta, t)$$

Thus, in view of equation (9b) and the postulated relation between temperature and heat release, for a vanishingly thin annulus,

$$\frac{\partial p'_c}{\partial r}(r_1, \theta, t) - \frac{\partial p'_w}{\partial r}(r_1, \theta, t) = K \frac{\partial p'_w}{\partial t}(r_1, \theta, t) \quad (11b)$$

where K is a quantity which depends on the combustion process within the annulus, and must in general vary with x . If the annulus has an axial extent much greater than the combustor diameter, then K may be regarded as constant. (This condition is not likely to be satisfied in a real combustor, but is considered to be qualitatively permissible.)

Equations (11a) and (11b) (K regarded as constant) are boundary conditions to be imposed on the oscillation in the surrounding gases. This oscillation is considered as composed of two parts, applicable for $r < r_1$ and $r > r_1$ separately, each part having the form of equation (10a). Thus, two sets of constants appear: α_c, β_c and α_w, β_w . Further boundary conditions specify vanishing normal velocity at the outer wall and at the centerbody:

$$\frac{\partial p'_w}{\partial r}(R, \theta, t) = 0 \quad (11c)$$

$$\frac{\partial p'_c}{\partial r}(r_0, \theta, t) = 0 \quad (11d)$$

The four homogeneous boundary conditions (eqs. (11)) relate the four constants $\alpha_c, \beta_c, \alpha_w$, and β_w , and yield the following determinantal equation for ω :

$$\begin{vmatrix}
 J_n\left(\frac{\omega r_1}{a_f}\right) & N_n\left(\frac{\omega r_1}{a_f}\right) & -J_n\left(\frac{\omega r_1}{a_f}\right) & -N_n\left(\frac{\omega r_1}{a_f}\right) \\
 -\left[\frac{\omega}{a_f} J'_n\left(\frac{\omega r_1}{a_f}\right) + iK\omega J_n\left(\frac{\omega r_1}{a_f}\right)\right] & -\left[\frac{\omega}{a_f} N'_n\left(\frac{\omega r_1}{a_f}\right) + iK\omega N_n\left(\frac{\omega r_1}{a_f}\right)\right] & \frac{\omega}{a_f} J'_n\left(\frac{\omega r_1}{a_f}\right) & \frac{\omega}{a_f} N'_n\left(\frac{\omega r_1}{a_f}\right) \\
 J'_n\left(\frac{\omega R}{a_f}\right) & N'_n\left(\frac{\omega R}{a_f}\right) & 0 & 0 \\
 0 & 0 & J'_n\left(\frac{\omega r_0}{a_f}\right) & N'_n\left(\frac{\omega r_0}{a_f}\right)
 \end{vmatrix} = 0 \quad (12)$$

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$$\frac{J'_n(Z)}{N'_n(Z)} = \frac{J'_n(z_0)}{N'_n(z_0)}$$

where $z \equiv \omega r/a_f$, $Z \equiv \omega R/a_f$, and $z_0 \equiv \omega r_0/a_f$. First-order terms in K define η and yield the amplification, which is equal to the negative of the imaginary part of ω :

$$-\frac{4R^2}{\pi r_1 K a_f^2} \text{Im}(\omega) = \frac{Z^2 \left(1 - \frac{z_0}{Z}\right)^2 \left[J_n(z_1) - N_n(z_1) \frac{J'_n(Z)}{N'_n(Z)} \right]^2}{\frac{1 - \left(\frac{n}{Z}\right)^2}{N_n'^2(Z)} - \frac{1 - \left(\frac{n}{z_0}\right)^2}{N_n'^2(z_0)}} \equiv A_1 \quad (13)$$

The grouping of constants on the left should be held constant for comparison of the effect of changes in flame-holder and centerbody configuration. In particular, since the quantity K (eq. (11b)) must depend on the rate of heat release per unit surface area of the annulus, $\pi r_1 K$ is proportional to the total rate of heat release at the annulus per unit length, and hence, for comparison purposes, should be held constant.

If there is no centerbody,

$$-\frac{4R^2}{\pi r_1 K a_f^2} \text{Im}(\omega) = Z^2 \frac{N_n'^2(Z) J_n^2(z_1)}{1 - \left(\frac{n}{Z}\right)^2} \equiv A_2 \quad (14)$$

The amplification A_2 , for the case of no centerbody, is plotted in figure 3 for several modes as a function of r/R . The greatest amplification occurs with the radial mode ($n = 0, m = 1$) when the burning is concentrated near the axis. Amplification is generally greatest when energy is added near a pressure loop (ref. 2, p. 226), and only the radial modes ($n = 0$) have such a loop near the axis. The higher modes ($n \geq 1$) have pressure nodes at the axis and are not amplified at all when burning is concentrated there. If the annulus is placed near the wall, higher modes (which have pressure loops at the wall) undergo the greater amplification.

The effect of a centerbody is shown in figure 4. Only the first sloshing mode ($n = 1, m = 0$) is considered. If the annulus is near the outer wall, the presence of a centerbody decreases the amplification somewhat, whereas the opposite is true if the annulus is near the center of the chamber.

Time Lag

Figures 3 and 4 predict the occurrence of the radial mode for burning near the axis or higher-order transverse modes for burning far from the axis. However, it is reasonable to inquire whether or not these results might be altered if there is introduced a time lag between state changes and fluctuations of the combustion process. The simplest lag hypothesis for the present problem is that a change in temperature produces a corresponding change in heat release at a somewhat later time. (Crocco and Cheng (ref. 5) and Cheng (ref. 4) consider the effects of time lag in their analyses of resonance in rocket engines.) The basic cause of such a lag is not specified, but might derive from the combustion kinetics of each burning element.

Thus, equation (11b) is replaced by

$$\frac{\partial p'_c}{\partial r}(r_1, \theta, t) - \frac{\partial p'_w}{\partial r}(r_1, \theta, t) = K \frac{\partial p'_w}{\partial t}(r_1, \theta, t - \tau) \quad (15)$$

where τ is a (constant) time lag. The analysis subsequent to equation (11b) may then be repeated, resulting in a modification of equations (13) and (14)

$$-\frac{4R^2}{\pi r_1 K a_f^2} \text{Im}(\omega) = [A_1 \text{ or } A_2] \cos \Omega \tau \quad (16)$$

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If $\Omega\tau < \pi$, increasing the frequency decreases the amplification and, in fact, causes damping if $\pi/2 < \Omega\tau < \pi$. Therefore, the amplification of the first sloshing mode ($n = 1, m = 0$) would be more intense relative to the other modes than is indicated in figures 3 and 4.

Actually, it would not be expected that a single time lag τ characterizes the entire burning process, but rather that each burning element would have a different response lag, so that each of the quantities $A \cos \Omega\tau$ in equation (16) is replaced by

$$A \int_0^{\infty} F(\tau) \cos \Omega\tau \, d\tau$$

where $F(\tau)$ is a distribution function for which a suitable choice might be

$$F(\tau) = \frac{1}{\tau^{*2}} \tau e^{-\tau/\tau^*} \quad (17)$$

which has the properties $F(0) = F(\infty) = 0$, $\int_0^{\infty} F(\tau) d\tau = 1$, and which has a maximum at $\tau = \tau^*$. Use of the preceding relations yields, in place of equation (16),

$$-\frac{4R^2}{\pi r_1^2 Ka_f} \text{Im}(\omega) = [A_1 \text{ or } A_2] \frac{1 - (\Omega\tau^*)^2}{[1 + (\Omega\tau^*)^2]^2} \quad (18)$$

Therefore, modes for which $\Omega\tau^* > 1$ would be damped if this equation holds. For amplified modes, the coefficient of the A's diminishes the amplification of modes of higher frequency.

FINITE TRANSVERSE WAVES

In one-dimensional unsteady flow, strong compression waves progressively steepen into shocks, which move with a speed greater than that of sound. This dependence of propagation speed on amplitude implies a dependence of wave form and frequency on amplitude in any one-dimensional cyclic process involving compression waves. The subsequent analysis is concerned with determining whether or not amplitude is similarly important for strong transverse oscillations in a cylinder, in order to indicate whether agreement between acoustic theory and experiment may be anticipated.

Differential Equations

An isentropic irrotational disturbance field (not necessarily weak) obeys the equations

$$\left. \begin{aligned} \text{Momentum:} \quad & \phi_t + \frac{q^2}{2} + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \frac{\bar{a}^2}{\gamma - 1} \\ \text{Continuity:} \quad & \rho_t + \nabla \cdot (\rho \underline{q}) = 0 \\ \text{State isentropy:} \quad & p \propto \rho^\gamma \\ \text{Definition of potential:} \quad & \underline{q} \equiv \nabla \phi \end{aligned} \right\} \quad (19)$$

where \bar{a} is the speed of sound in the absence of any oscillation. Equations (19) may be combined to yield a differential equation for ϕ :

$$\phi_{tt} - \bar{a}^2 \nabla^2 \phi = - \left[(q^2)_t + \frac{1}{2} \underline{q} \cdot \nabla (q^2) + (\gamma - 1) \left(\phi_t + \frac{1}{2} q^2 \right) \nabla^2 \phi \right] \quad (20)$$

Next, a finite-strength wave pattern is sought in the form of a power series in an amplitude parameter ϵ

$$\phi \equiv \bar{a}R \left(\epsilon \phi^{(1)} + \epsilon^2 \phi^{(2)} + \dots \right) \quad (21)$$

Substitution into equation (20) and collection of terms of first and second order in ϵ yield, respectively,

$$\phi_{tt}^{(1)} - \bar{a}^2 \nabla^2 \phi^{(1)} = 0 \quad (22)$$

$$\phi_{tt}^{(2)} - \bar{a}^2 \nabla^2 \phi^{(2)} = - 2\bar{a}R \left(\nabla \phi^{(1)} \cdot \nabla \phi_t^{(1)} + \frac{\gamma - 1}{2\bar{a}^2} \phi_t^{(1)} \phi_{tt}^{(1)} \right) \quad (23)$$

and so forth.

Solutions

One-dimensional motion. - A weak traveling wave may have the form

$$\phi^{(1)} = k \sin \Omega \left(t + \frac{x}{a} \right) \quad (24)$$

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which satisfies equation (22). Then equation (23) may be written

$$\phi_{tt}^{(2)} - \frac{\gamma-2}{a^2} \phi_{xx}^{(2)} = \frac{\gamma+1}{2} \frac{\Omega^3 R}{a} k^2 \sin 2\Omega \left(t + \frac{x}{a} \right) \quad (25)$$

No solution of equation (25) can be written consistent with the existence of a strong wave moving at constant speed with a permanent wave form, because the solution must contain a particular integral of the type

$$- \frac{\gamma+1}{8} \frac{\Omega R}{a} k^2 \Omega t \cos 2\Omega \left(t + \frac{x}{a} \right) \quad (26)$$

Therefore, no permanent one-dimensional finite wave is possible (ref. 2, p. 480).

Transverse motion in a cylindrical pipe. - From equation (2), the first sloshing mode ($n = 1, m = 0$) is chosen for study:

$$\phi^{(1)} = J_1(z) \sin (\Omega t + \theta) \quad (27)$$

Equation (27) represents a wave spinning in a clockwise direction. The addition of a wave spinning in the opposite direction would yield a standing, or sloshing, motion. Equation (27) cannot be interpreted as representing a succession of pulses traveling with the speed of sound, as can equation (24). The "wave front" of equation (27) travels at an apparent Mach number varying from zero at the cylinder axis to 1.84 at the cylinder wall and therefore should be regarded essentially as an interference pattern rather than as a wave front. The foregoing consideration suggests that the progressive steepening with time, characteristic of isolated pulses and illustrated by equation (26), need not necessarily be expected in the case of transverse motion in a cylinder. The attempt is therefore made to obtain $\phi^{(2)}$ in the form

$$\phi^{(2)} = f(z) \sin 2(\Omega t + \theta) \quad (28)$$

Substituting equations (27) and (28) into equation (23) yields an equation for $f(z)$

$$f'' + \frac{f'}{z} + 4 \left(1 - \frac{1}{z^2} \right) f = z \left[J_1'^2(z) - \left(\frac{\gamma-1}{2} + \frac{1}{z^2} \right) J_1^2(z) \right] \quad (29)$$

The boundary condition at the cylinder wall is

$$f'(Z) = 0 \quad (30)$$

and, at $z = 0$, $f(z)$ must be an analytic function.

The solution to equation (29) may be written as

$$f(z) = \frac{z}{2} \left\{ (\gamma + 1) \left[DJ_2(2z) - g(z) \right] + J_1^2(z) \right\} \quad (31)$$

where, by substitution into equation (29), $g(z)$ is a particular solution of the equation

$$g'' + \frac{1}{z} g' + 4 \left(1 - \frac{1}{z^2} \right) g = J_1^2(z) \quad (32)$$

and $DJ_2(2z)$ is the complementary solution of the same equation. The constant D is chosen so that boundary condition (30) is satisfied. Equation (32) may be solved by a series method, adopted for convenience in computation, yielding the result

$$g = \frac{1}{3} \left(\frac{z}{2} \right)^4 - \frac{7}{24} \left(\frac{z}{2} \right)^6 + \frac{19}{180} \left(\frac{z}{2} \right)^8 - \frac{187}{8640} \left(\frac{z}{2} \right)^{10} + \frac{437}{151200} \left(\frac{z}{2} \right)^{12} - \frac{1979}{7257600} \left(\frac{z}{2} \right)^{14} + \frac{4387}{228614400} \left(\frac{z}{2} \right)^{16} - \dots \quad (33)$$

Introduction of equations (31) and (33) into boundary condition (30) gives

$$D = -0.384433 \quad (34)$$

Equations (28), (31), (33), and (34) complete the solution, indicating that a "permanent" transverse acoustic wave of finite amplitude is possible, at least to second order. Thus, a strong wave form may have a Fourier series, consisting of the linearized solution and its harmonics, for transverse motion in a cylinder, but not for a purely one-dimensional wave.

Numerical Results and Discussion

The results of the previous section may be expressed through the equations of motion (eqs. (19)) in terms of pressure fluctuation:

$$\frac{p}{(p)_{\epsilon^*=0}} = 1 + \epsilon^* P_1 \cos(\Omega t + \theta) + \epsilon^{*2} [P_{20} + P_{22} \cos 2(\Omega t + \theta)] + \dots \quad (35)$$

where the definition

$$\epsilon \equiv - \frac{\epsilon^*}{rZJ_1(Z)} \quad (36)$$

is made, so that $P_1(Z) = 1$, as a matter of choice. Furthermore,

$$\left. \begin{aligned} P_1 &= \frac{J_1(z)}{J_1(Z)} \\ P_{20} &= \frac{1}{4\gamma} \left[\left(1 - \frac{1}{z^2} \right) \frac{J_1^2(z)}{J_1^2(Z)} - \frac{J_1'^2(z)}{J_1'^2(Z)} \right] \\ P_{22} &= \frac{1}{4\gamma} \left\{ 4(\gamma + 1) \left[\frac{g(z)}{J_1^2(Z)} - c \frac{J_2(2z)}{J_1^2(Z)} \right] - \left[\left(3 + \frac{1}{z^2} \right) \frac{J_1^2(z)}{J_1^2(Z)} - \frac{J_1'^2(z)}{J_1'^2(Z)} \right] \right\} \end{aligned} \right\} \quad (37)$$

With $\gamma = 1.4$ assumed, numerical results for P_1 , P_{20} , and P_{22} are shown in figure 5 as functions of r/R (equivalent to z/Z). The symbol P_1 represents the radial distribution of pressure amplitude in a spinning acoustic wave of the first sloshing mode ($n = 1$, $m = 0$); P_{20} is a steady second-order correction to the radial pressure distribution, and signifies that the oscillation tends to exhaust the center of the cylinder and compress the fluid near the wall, in the manner of a centrifuge; and P_{22} is the radial distribution of pressure amplitude of the first overtone. This overtone is strongest relative to the fundamental tone at the wall and is relatively weakest at the cylinder axis.

The modification of the wave form, when the oscillation is rather weak but not of infinitesimal strength, is illustrated in figure 6. Owing to terms of second order in ϵ^* , the pressure crests at the cylinder wall are steeper and higher than those of the fundamental harmonic wave, and the troughs are broader and shallower, in the familiar pattern of second-harmonic distortion. A lesser second-order effect is the slight increase in average pressure at the cylinder wall, owing to P_{20} .

The steepness of the crests illustrated in figure 6 is suggestive of the steepening of one-dimensional compression waves leading to shock formation, but is not a progressive steepening with time as in the one-dimensional case. Rather, the modified wave shown in figure 6 may be imagined to proceed around the cylinder wall without change ad infinitum.

To the present order of approximation, the fundamental frequency does not change either with time or with amplitude, although overtones appear in proportion to amplitude. It may therefore be concluded that violent combustion-chamber resonance is not necessarily incompatible with acoustic-theory predictions.

The permanent isentropic character of the solution obtained in the foregoing analysis suggests that strong transverse oscillations may not be subject to the dissipative normal stresses which tend to limit the amplitude of a one-dimensional strong-wave cycle involving shock waves.

CONCLUDING REMARKS

On the assumption that combustion-chamber resonance results from the amplification of an initially weak wave of transverse type coupled with a pressure-dependent rate of energy release in the combustor, three problems were discussed:

1. Modes of oscillation: It was shown that if the mean flow Mach number is assumed very small, then the possible modes of weak disturbance are governed by a wave equation in classical form except for a term depending on the streamwise variation of temperature in the combustor. Two such modes were determined by numerical integration, subject to the assumption that in the afterportion of the combustor the oscillation is purely transverse. These modes conform to the classical spiral modes (if the wave is a traveling one) upstream of the burning zone, with a pitch depending on the ratio of chamber diameter to length of burning zone, as well as on the combustor temperature.

2. Amplification at a thin annulus: In the approximation of very small Mach number, the burning process does not amplify the modes described. The especially vigorous burning in a flame-holder wake would provide amplification, however. A flame-holder wake is idealized as a thin annulus performing work upon the surrounding gases in proportion to the instantaneous pressure level.

Analysis of the resulting amplification indicates that concentration of vigorous burning near the center of the combustor amplifies the radial mode most strongly. Burning concentrated near the combustor wall amplifies the higher sloshing modes. The presence of a centerbody slightly increases (decreases) the amplification of the sloshing mode if energy is added near the inner (outer) wall.

Supposition of a time lag between temperature disturbance and heat-release rate in the annulus favors amplification of lower-frequency modes; in particular, the first sloshing mode, which has the lowest frequency of all possible modes.

3. Finite transverse waves: The nature of strong transverse acoustic oscillations was considered. It was shown that, to second order, amplitude affects wave form in a spinning oscillation, causing the pressure crests to be higher and sharper and the troughs shallower and broader. The average pressure is diminished near the axis and increased near the wall, suggesting a sort of centrifugal pumping. The wave form does not undergo progressive change leading to formation of a shock, as in one-dimensional flow; rather, the fundamental frequency does not depend on amplitude, to the present order of approximation.

Furthermore, the permanently isentropic nature of the transverse-wave solution suggests that transverse modes in a cylindrical enclosure are not inhibited by viscous effects to the same degree as are one-dimensional waves, and for this reason are perhaps better able to attain extreme amplitudes.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, May 19, 1954

APPENDIX - NOTATION

The following notation is used in this report:

A_1, A_2	amplification rates (eqs. (13) and (14))
\bar{a}	speed of sound, $\sqrt{(\gamma - 1)c_p \bar{T}}$
a_f	final speed of sound attained in combustor
C	constant (eqs. (2) and (10))
c_v	specific heat at constant volume
c_p	specific heat at constant pressure
D	constant (eq. (34))
f	function associated with second-order solution (eq. (28))
G	function used in description of mode (eqs. (10))
g	function associated with second-order solution (eq. (31))
H	steady rate of heat release per unit volume
Im	imaginary part
J_n	Bessel function of first kind of order n (ref. 6)
K	constant of proportionality (eq. (11b))
k	amplitude (eq. (24))
L	axial extent of region of substantial temperature change (fig. 2)
\bar{M}	mean flow Mach number
m	each of the sequence of numbers for which $J'_n = 0$
n	mode
N_n	Bessel function of second kind of order n (ref. 6)
P_1, P_{20}, P_{22}	pressure coefficients (eqs. (37))
p	static pressure

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q	magnitude of disturbance velocity vector
\underline{q}	disturbance velocity vector
R	inside radius of combustion chamber (fig. 1)
r	radial coordinate (fig. 1)
r_0, r_1	combustor dimensions (fig. 1)
T	static temperature
t	time
u	axial perturbation velocity
x	axial coordinate (sketch (a))
Z	dimensionless quantity, $\frac{\Omega R}{a_f}$
z	dimensionless coordinate, $\frac{\Omega r}{a_f}$
z_0, z_1	$\frac{\Omega r_0}{a_f}, \frac{\Omega r_1}{a_f}$
α	coefficient (eq. (10a))
β	coefficient (eq. (10a))
γ	ratio of specific heats, c_p/c_v
ϵ, ϵ^*	amplitude parameters (eqs. (21) and (36), respectively)
η	coefficient involved in solution of equation (12)
θ	angular coordinate (fig. 1)
ρ	density
τ	time lag (eq. (15))
τ^*	most probable time lag (eq. (17))
ϕ	disturbance velocity potential ($\underline{q} = \nabla \phi$)
Ω	frequency in absence of burning
ω	frequency

Superscripts:

- (1) first-order solution
- (2) second-order solution
- ' disturbance quantities and, when convenient, ordinary differentiation

Barred symbols denote flow quantities in absence of disturbance.

Subscripts:

- c region $r_0 < r < r_1$
- n mode
- w region $r_1 < r < R$

Subscript notation for partial differentiation when convenient.

REFERENCES

1. Morse, Philip M.: Vibration and Sound. Second ed., McGraw-Hill Book Co., Inc., 1948, par. 32.
2. Rayleigh: The Theory of Sound. Vol. II. Dover Pub., 1945.
3. Grad, Harold: Resonance Burning in Rocket Motors. Communications on Pure and Appl. Math., vol. II, no. 1, Mar. 1949, pp. 79-102.
4. Cheng, Sin-I.: High Frequency Combustion Instability in Solid Propellant Rockets. Rep. No. 221, Aero. Eng. Lab., Princeton Univ., Feb. 1953.
5. Crocco, Luigi, and Cheng, Sin-I.: High Frequency Combustion Instability in Rocket Motor with Concentrated Combustion. Jour. Am. Rocket Soc., vol. 23, no. 5, Sept.-Oct. 1953, pp. 301-313.
6. Jahnke, Eugen, and Emde, Fritz: Tables of Functions. Fourth ed., Dover Pub., 1945.

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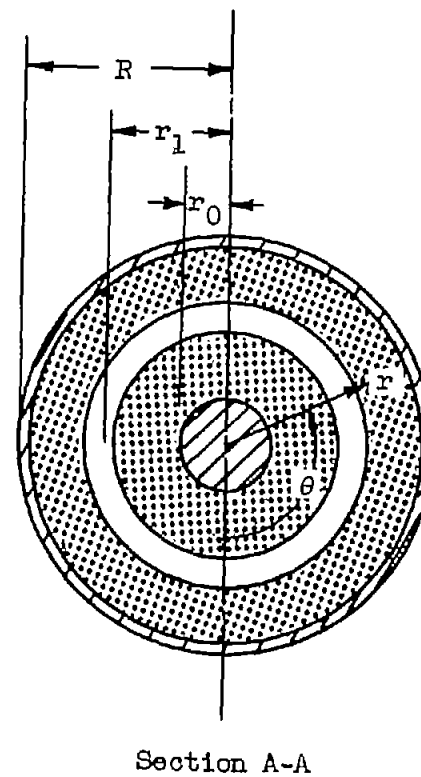
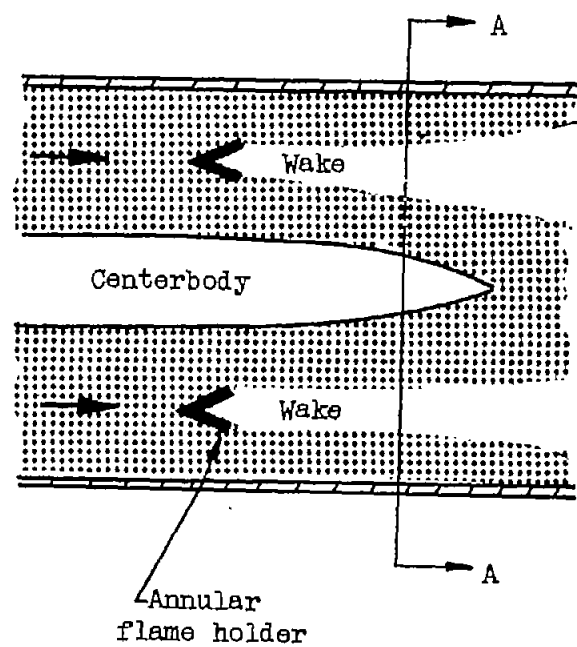


Figure 1. - Assumed configuration of combustor.

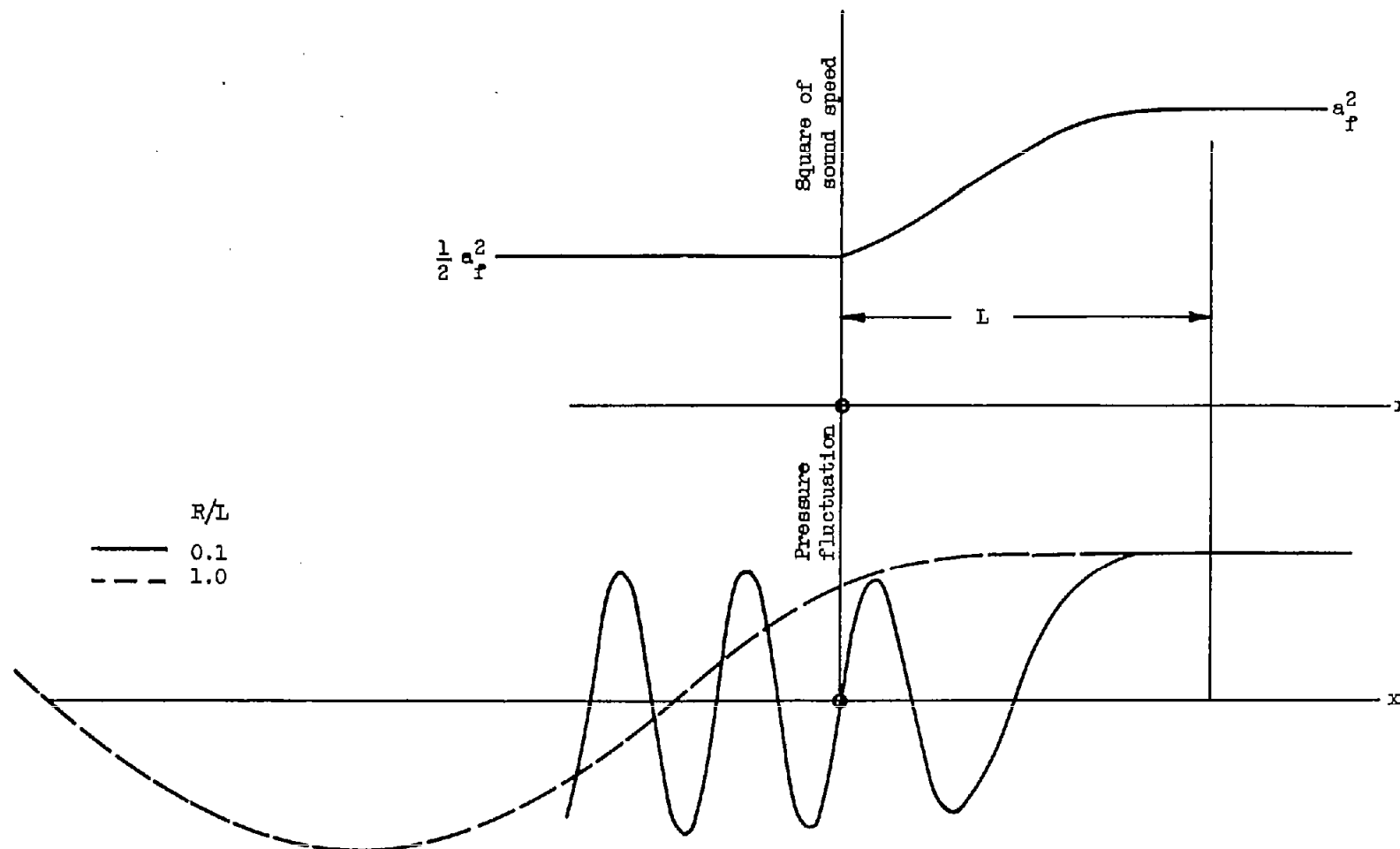


Figure 2. - Longitudinal pressure variation for a combined mode (defined by eqs. 10(a) and (d) for $n = 1$, $m = 0$) for a particular axial distribution of sound speed. $M \ll 1$.

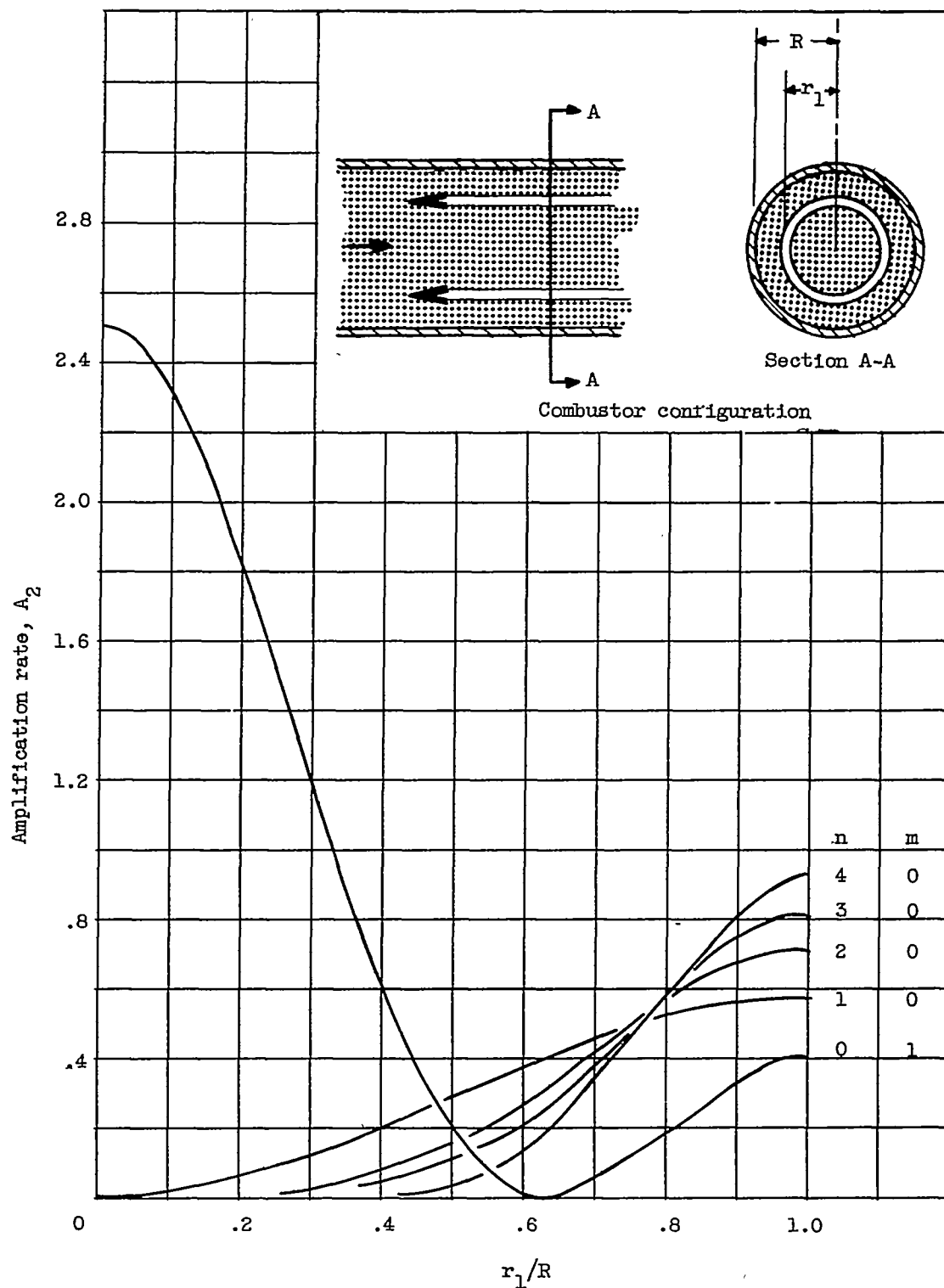


Figure 3. - Amplification rate for various modes. Burning in narrow annulus.

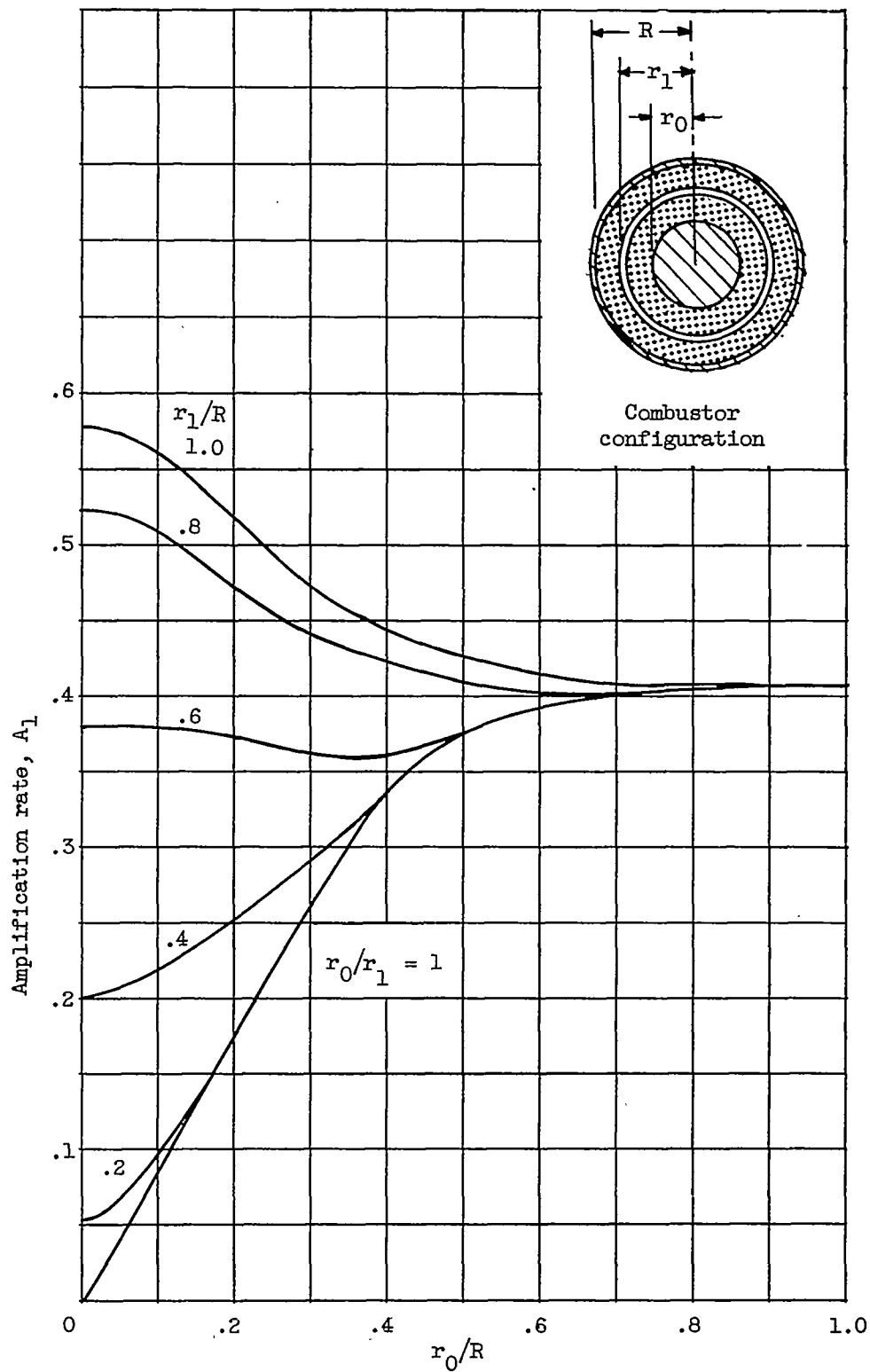


Figure 4. - Amplification rate for first sloshing mode ($n = 1$, $m = 0$) with centerbody and burning in annulus.

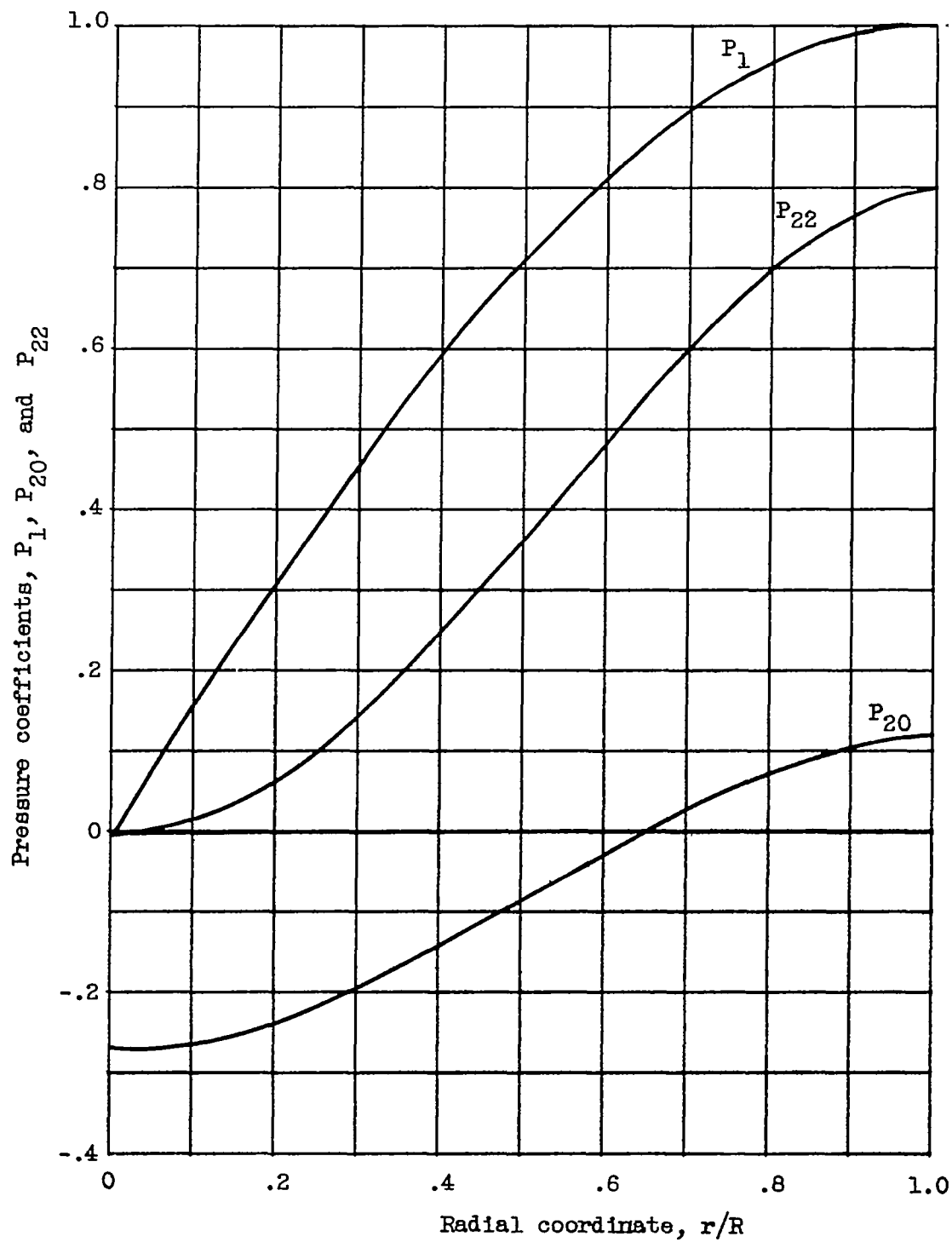


Figure 5. - Radial distribution of pressure amplitude.

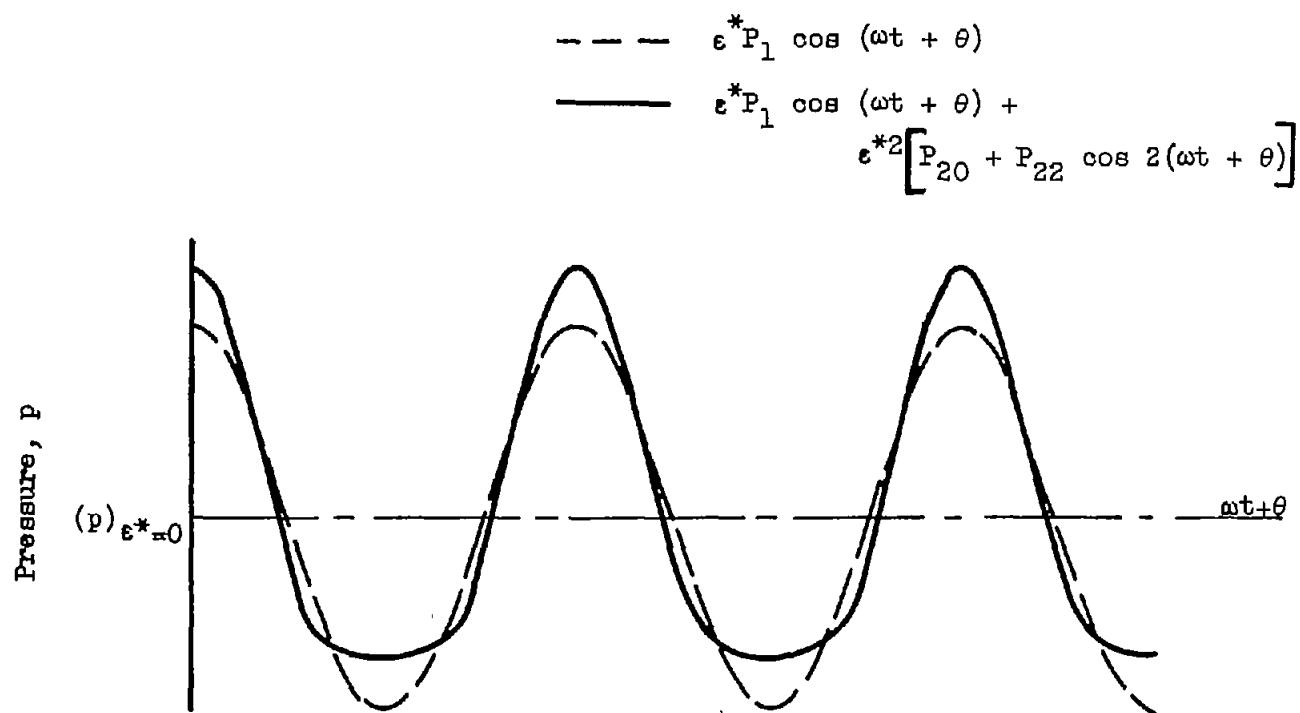


Figure 6. - Modification of fundamental pressure fluctuation at cylinder wall, owing to second-order terms.